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Null controllability of a system of coupled and degenerate semilinear parabolic equations with first-order terms

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Abstract

In this paper, we present a null controllability result for a coupled system of degenerate semilinear parabolic equations with first-order terms. The main result is to obtain a new Carleman-type inequality for the coupled system in a first step. Then, from this inequality, we prove a null controllability result for the system of degenerate linear parabolic equations with first order terms. And with Kakutani's fixed point theorem, we show that the initial system is null controllable.

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1 Introduction

In this paper we are interested in null controllability properties of a degenerate semilinear parabolic equation. We consider $k \in \mathcal{C}([0,1])$, $k > 0$ in $(0,1]$, $k(0) = 0$. Let us fix $T > 0$ and a

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non-empty open subset $\omega \subset (0,1)$. The degenerate parabolic equations that we want to analyze are given by the following system

$$\begin{cases} y_t - (k(x)y_x)_x + F(x,t,y,y_x) = h1_\omega & \text{in } Q, \\ z_t - (k(x)z_x)_x + G(x,t,z,z_x) = y1_{\mathcal{O}} & \text{in } Q, \\ y(\sigma,t) = z(\sigma,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) \;\; z(x,0) = z_0(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

where $Q = \Omega \times (0,T)$, $\Omega = (0,1)$, $\Sigma = \{0,1\} \times (0,T)$, the initial data $y_0,\ z_0$ are given in $L^2(\Omega)$, $h \in L^2(Q)$ is a controls, k represents the dispersion coefficient and F and G are globally Lipschitz functions. Here, $\omega \subset \Omega$ is small open control set, $\mathcal{O} \subset \Omega$ is an small open coupling set and 1_ω and $1_{\mathcal{O}}$ denote the characteristic functions of ω and $\mathcal{O}$ respectively.

We consider that the problem is weakly degenerate, which amounts to considering the following conditions on the coefficient k.

$$\begin{gathered} k \in \mathcal{C}([0,1]) \cap \mathcal{C}^1((0,1]),\ k > 0 \text{ in } (0,1] \text{ and } k(0) = 0, \\ \text{there exists } \lambda \in [0,1) \text{ such that } xk'(x) \le \lambda k(x) \text{ for all } x \in [0,1]. \end{gathered} \tag{1.2}$$

Remark 1.1. *Notice that, under these assumptions, $x^\lambda/k(x)$ is not decreasing and then, since $0 \le \lambda < 1, \dfrac{1}{\sqrt{k}}, \dfrac{1}{k} \in L^1(0,1)$.*

Our aim is to give conditions on f in such a way that the system (1.1) is null controllable. That is, give H a Hilbert space such that for any $y_0, z_0 \in H$ it exists $h \in L^2(q)$ $(q = \omega \times (0,T))$ such that the corresponding solution to (1.1) satisfies

$$y(.,T) = z(.,T) = 0. \tag{1.3}$$

Next, we need to introduce hypotheses about the functions F and G. To do this, we consider a function $\beta_i(x)$ such as

Hypotheses 1. *Let $F, G : [0,1] \times [0,T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be such that*

i) $F(x,t;0,0) = G(x,t;0,0) = 0, \quad \forall (x,t) \in Q.$

ii) $F(\cdot,\cdot,s,p), G(\cdot,\cdot,s,p) \in L^\infty(Q), \quad \forall (s,p) \in \mathbb{R}^2$.

iii) *$F(x,t,\cdot,\cdot), G(x,t,\cdot,\cdot)$ are globally Lipschitz for every $(x,t) \in Q$ with Lipschitz constant independent of (x,t).*

iv) *$F(\cdot,\cdot,s,p) = f_1(\cdot,\cdot,s,p) + f_2(\cdot,\cdot,s,p)p$ for every $(s,p) \in \mathbb{R}^2$, where $f_2(\cdot,\cdot,s,p) \in L^\infty(Q)$, for every $(s,p) \in \mathbb{R}^2$ and $\left|\dfrac{f_2(x,t,s,p)}{\beta_1(x)}\right| \le C$ almost everywhere $(x,t) \in Q$, and almost everywhere $(s,p) \in \mathbb{R}^2$.*

$G(\cdot,\cdot,s,p) = g_1(\cdot,\cdot,s,p) + g_2(\cdot,\cdot,s,p)p$ for every $(s,p) \in \mathbb{R}^2$, where $f_2(\cdot,\cdot,s,p) \in L^\infty(Q)$, for every $(s,p) \in \mathbb{R}^2$ and $\left|\dfrac{g_2(x,t,s,p)}{\beta_2(x)}\right| \le C$ almost everywhere $(x,t) \in Q$, and almost everywhere $(s,p) \in \mathbb{R}^2$.

In recent years, the study of the null controllability properties of degenerate parabolic equations has been intensive. See for example [1], [14], [10], [3] and [9]. In these papers, the authors studied the null controllability properties of linear degenerate one-dimensional equations without first-order terms. That is, F is of the form $F(x,t,y,y_x) = b(x,t)y$. In [7], the authors studied the null controllability properties when $F(x,t,y,y_x) = b(x,t)y + c(x)y_x$. It should also be noted that systems of coupled degenerate equations have been the subject of numerous studies [5, 6]. In [16], the authors studied a degenerate coupled system with $k(x) = x^\alpha$. The results we present in this paper are a generalization of those presented in [12] but applied to a system of semilinear degenerate coupled equations with first-order terms. To present our results, we must introduce assumptions about the functions F and G. To do this, we fix a function $\beta_i(x)$ $(i = 1,2)$ such that

$$\frac{\beta_i(x)}{x} \in L^\infty(0,1), \qquad \text{for } i = 1,2. \tag{1.4}$$

Remark 1.2. *Under the conditions of the coefficient k, and using (1.4), we deduce the following inequality*

$$\frac{\beta_i^2(x)}{k(x)} \leq \frac{C(\beta_i)}{k(1)}, \quad a.e.\ x \in \Omega \quad \text{and for } i = 1,2. \tag{1.5}$$

The main result of this paper is:

Theorem 1.3. *For $T > 0$ and $y_0, z_0 \in L^2(0,1)$. Let us assume (1.2) and (1.4) and assumptions (1). Then, system (1.1) is null controllable, that is, there exists $h \in L^2(\omega \times (0,T))$ and for all $y_0, z_0 \in L^2(\Omega)$, such that the solution pair (y,z) of the system (1.1) satisfies $y(x,T) = y(x,T) = 0$ for all $x \in [0,1]$. Furthermore, there exists a constant $C > 0$ depending only on T, such that*

$$\int_0^1 \int_\omega |h|^2 \mathrm{d}x\mathrm{d}t \leq C\left(\|y_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)}\right). \tag{1.6}$$

The aim of this paper is to prove Theorem 1.3. To this end, we study the system of linearized degenerate coupled parabolic equations and, with a fixed point argument, we obtain our main result. The rest of the paper is organized as follows. In the next, we present the workspaces, the degenerate coupled linearized system. We conclude this section with an existence and uniqueness of solution result for the linearized system. In Section 3, we present and study the coupled and degenerate linear problem associated with the initial problem. We prove a new Carleman inequality for the adjoint coupled system associated with the linear one. In Section 4, we prove our main result.

2 Preliminaries

In this section we study the null controllability of the degenerate linear equation

$$\begin{cases} y_t - (k(x)y_x)_x + b_1(x,t)y + \beta_1(x)c_1(x,t)y_x = h1_\omega & \text{in } Q, \\ z_t - (k(x)z_x)_x + b_2(x,t)z + \beta_2(x)c_2(x,t)z_x = y1_\mathcal{O} & \text{in } Q, \\ y(\sigma,t) = z(\sigma,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) \ \ z(x,0) = z_0(x) & \text{in } \Omega. \end{cases} \tag{2.1}$$

Now, we introduce the following Sobolev spaces

$$\begin{aligned} H^1_k(\Omega) &= \{u \in L^2(\Omega);\ u \text{ absolute continuous in } [0,1],\ \sqrt{k}u_x \in L^2(\Omega),\ u(0)=u(1)=0\}, \\ H^2_k(\Omega) &= \{u \in H^1_k(\Omega);\ k(x)u_x \in H^1(\Omega)\}, \end{aligned}$$

with respectively the following norms :

$$\begin{aligned} \|u\|^2_{H^1_k(\Omega)} &= \|u\|^2_{L^2(\Omega)} + \|\sqrt{k}u_x\|^2_{L^2(\Omega)}, \quad \forall u \in H^1_k(\Omega); \\ \|u\|^2_{H^2_k(\Omega)} &= \|u\|^2_{H^1_k(\Omega)} + \|(ku_x)_x\|^2_{L^2(\Omega)}, \quad \forall u \in H^2_k(\Omega). \end{aligned}$$

Consider the unbounded operator $A : D(A) = H^2_k(\Omega) \to L^2(\Omega)$ defined by $Au = (k(x)u_x)_x$, $u \in D(A)$. Recall that the operator $Au = (ku_x)_x$, $u \in D(A) = H^2_k(\Omega) \subset H^1_k(\Omega)$, is a closed , symmetric, self-adjoint and negative operator in addition, $D(A)$ is dense in $L^2(\Omega)$ (see [1, 8, 9, 10]). Moreover, it is infinitesimal generator of a strongly continuous semi-group (see [15, 17]). Notice that the following existence and uniqueness results of solutions of the system (2.1) is well known. So, this permits to deduce the following result (see e.g. [8, 5]),

Theorem 2.1. *Suppose that $b_i, c_i \in L^\infty(Q)$, under the hypothesis (1.2), β_i satisfies* (1.4) *for $i = 1, 2$, $h \in L^2(q)$ and $y_0, z_0 \in L^2(0,1)$, the system (2.1) admits a unique solution $y, z \in X = \mathcal{C}([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_k(\Omega))$ and we have :*

$$\sup_{[0,T]} \|(y(t), z(t)\|^2_{(L^2(\Omega))^2} + \int_0^T \|y(t), z(t)\|^2_{(H^1_k(\Omega))^2} \mathrm{d}t \le C \left(\|h\|^2_{L^2(q)} + \|(y_0, z_0)\|^2_{(L^2(\Omega))^2} \right). \quad (2.2)$$

If moreover $y_0, z_0 \in D(A)$ then, $y \in X_1 = H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2_k(\Omega)) \cap \mathcal{C}([0,T]; H^1_k(\Omega))$ and

$$\begin{aligned} \sup_{[0,T]} \|(y(t), z(t)\|^2_{(H^1_k(\Omega))^2} &+ \int_0^T \left(\|(y_t(t), z_t(t)\|^2_{(L^2(\Omega))^2} + \|(ky_x)_x, (kz_x)_x\|^2_{(L^2(\Omega))^2} \right) \mathrm{d}t \le C\|h\|^2_{L^2(Q)} \\ &+ C\|(y_0, z_0)\|^2_{(H^1_k(\Omega))^2}. \end{aligned} \quad (2.3)$$

It is well known that the null controllability properties of system (2.1) can be characterized by its adjoint system:

$$\begin{cases} -w_t - (k(x)w_x)_x + b_1(x,t)w - (\beta_1(x)c_1(x,t)w)_x = v1_{\mathcal{O}} & \text{in } Q, \\ -v_t - (k(x)v_x)_x + b_2(x,t)v - (\beta_2(x)c_2(x,t)v)_x = 0 & \text{in } Q, \\ w(\sigma,t) = v(\sigma,t) = 0 & \text{on } \Sigma, \\ w(x,T) = w_T(x) \ \ v(x,T) = v_T(x) & \text{in } \Omega. \end{cases} \quad (2.4)$$

3 Linear coupled system

In this section we study the null controllability of the degenerate linear equation.

3.1 Carleman inequality for a degenerate system

The main result of this subsection is the following

Theorem 3.1. *We assume that $b_i, c_i \in L^\infty(Q)$ $(i = 1, 2)$, under the hypothesis (1.2), β_i satisfies (1.4) for $i = 1, 2$, that (1.2) holds true and $\omega \cap \mathcal{O} \neq \emptyset$. For any subset ω' such that $\omega' \subset\subset \omega \cap \mathcal{O}$, there exists two positive constants $C = C(\Omega, \omega, k, \|b_i\|_\infty, \|c_i\|_\infty,)$ and s_0 such that for every $s \geq s_0$ and for every solution of the system (2.4), satisfy the following inequality:*

$$\begin{aligned} & \int_Q \left(s\theta k(x)(|v_x|^2 + |w_x|^2) + s^3\theta^3 \frac{x^2}{k(x)}(|v|^2 + |w|^2) \right) e^{2s\Phi} \mathrm{d}x\mathrm{d}t \\ \leq \quad & C \int_0^T \int_{\omega'} |v|^2 \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.1}$$

Note that Carleman's Inequality (3.1) will allow us to establish the observability inequality, which is of the form

Proposition 3.2. *(Observability inequality) For any $T > 0$ and $s_0 \geq s$, there exists a constant $C > 0$ such that and for every solution of the system (2.1), satisfy the following inequality:*

$$\|w(.,0)\|^2_{L^2(\Omega)} + \|v(.,0)\|^2_{L^2(\Omega)} \leq C \int_0^T \int_{\omega'} |v(x,t)|^2 \mathrm{d}x\mathrm{d}t. \tag{3.2}$$

Moreover, we have

$$\|h\|_{L^2(\omega\times(0,T))} \leq C \left(\|y_0\|_{L^2(\Omega)} + \|z_0\|_{L^2(\Omega)} \right). \tag{3.3}$$

In order to give the proof of Carleman's inequality, we introduce some functions as follows : For $\omega = (a, b)$ let $\alpha = \dfrac{2a+b}{3}$, $\beta = \dfrac{a+2b}{3}$ and let $\rho \in \mathcal{C}^2(\mathbb{R})$ be such that $0 \leq \rho \leq 1$ and $\rho(x) = 1$ if $x \in (0, \alpha)$ and $\rho(x) = 0$ if $x \in (\beta, 1)$. Let us define $\theta(t) = \dfrac{1}{(t(T-t))^4}$ $\forall t \in (0, T)$, $\psi(x) = c_1 \left(\displaystyle\int_0^x \frac{r}{k(r)} \mathrm{d}r - c_2 \right)$ with $c_1 > 0$ and $c_2 > \dfrac{1}{k(1)(2-\lambda)}$. Let $\Psi(x) = e^{r\sigma(x)} - e^{2r\|\sigma\|_\infty}$ be where $r > 0$ and σ a function which satisfies : $\sigma \in \mathcal{C}^2([0,1])$, $\sigma > 0$ in Ω, $\sigma = 0$ on $\partial\Omega$ and $\sigma_x(x) \neq 0$ in $[0,1] \setminus \omega_0$ where ω_0 is a open set of Ω such that $\omega_0 \subset\subset \omega$.

Let us define $\Phi(x,t) = \theta(t)[\rho(x)\psi(x) + (1-\rho(x))\Psi(x)]$. The existence of the function σ is proved in [13].

Remark 3.3. *This shows that the observation of v on $\omega' \subset\subset \mathcal{O}$ controls both functions w and v throughout the domain, which is the goal of Carleman's inequality for coupled systems.*

Consider the following adjoint system where H_0, $H \in L^2(Q)$ and $w_T \in L^2(\Omega)$:

$$\begin{cases} w_t + (k(x)w_x)_x = H_0 + (\beta(x)H)_x & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x,T) = w_T(x) & \text{in } \Omega. \end{cases} \tag{3.4}$$

We have the following result

Theorem 3.4. *Under the hypothesis (1.2) and (1.4), for all $T > 0$. Then, there exists two positive constants then, there exists two constants $C > 0$ and $s_0 > 0$ such that for all $s \geq s_0$ and all solution w of the system (3.4), satisfies*

$$\begin{aligned} &\int_Q \left((s\theta)^{l+1}k(x)|w_x|^2 + (s\theta)^{3+l}\frac{x^2}{k(x)}|w|^2\right)e^{2s\Phi}\mathrm{d}x\mathrm{d}t \\ \leq\ & C\left(\int_Q (s\theta)^l|H_0|^2 + \int_Q (s\theta)^{3+l}\frac{\beta(x)^2}{k(x)}|H|^2 + \int_0^T\int_\omega (s\theta)^l|w|^2\right)e^{2s\Phi}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.5}$$

The proof of Theorem 3.4 is a consequence of the following result.
Consider the following adjoint system :

$$\begin{cases} w_t + (k(x)w_x)_x = H_1 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x,T) = w_T(x) & \text{in } \Omega. \end{cases} \tag{3.6}$$

We have

Proposition 3.5. *Under the hypothesis (1.2), and (1.4) for all $T > 0$ and $l \in \mathbb{N}$ then, there exists two constants $C > 0$ and $s_0 > 0$ such that for all $s \geq s_0$ and all solution w of the system (3.6), we have*

$$\begin{aligned} &\int_Q \left(s^{l+1}\theta^{l+1}k(x)|w_x|^2 + s^{l+3}\theta^{l+3}\frac{x^2}{k(x)}|w|^2\right)e^{2s\Phi}\mathrm{d}x\mathrm{d}t \\ \leq\ & C\left(\int_Q (s\theta)^l e^{2s\Phi}|H_1|^2\mathrm{d}x\mathrm{d}t + \int_0^T\int_\omega (s\theta)^l e^{2s\Phi}|w|^2\mathrm{d}x\mathrm{d}t\right). \end{aligned} \tag{3.7}$$

For the proof of Proposition 3.5 see [5]
The proof of Theorem 3.4 follows from using Assumption (1.4) and replacing in the inequality (3.7), H_1 by $H_1 = H_0 + (\beta(x)H)_x$. □

Proof of Theorem 3.1
Let $\omega' \subset\subset \omega \cap \mathcal{O}$. Let us apply Theorem 3.4 by taking $l = 0$ to the two equations of the system (2.4). By adding the two inequalities obtained, we find

$$\begin{aligned} &\int_Q \left(s\theta k(x)(|v_x|^2 + |w_x|^2) + s^3\theta^3\frac{x^2}{k(x)}(|v|^2 + |w|^2)\right)e^{2s\Phi}\mathrm{d}x\mathrm{d}t \\ \leq\ & C\int_Q \left((|b_1w + v1_{\mathcal{O}}|^2 + |b_2v|^2) + (s\theta)^3\frac{\beta(x)^2}{k(x)}(|(\beta_1c_1w)_x|^2 + |(\beta_2c_2v)_x|^2)\right)e^{2s\Phi}\mathrm{d}x\mathrm{d}t \\ +\ & C\int_0^T\int_{\omega'}\left(|v|^2 + |w|^2\right)e^{2s\Phi}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.8}$$

Thus, we have

$$\int_Q (|b_2v|^2 + |b_1w + v1_{\mathcal{O}}|^2)e^{2s\Phi}\mathrm{d}x\mathrm{d}t \leq C\int_Q |v|^2e^{2s\Phi} + C\int_Q (|w|^2 + |v|^2 1_{\mathcal{O}})e^{2s\Phi}\mathrm{d}x\mathrm{d}t. \tag{3.9}$$

$$\begin{aligned}\int_Q (s\theta)^3 \frac{\beta(x)^2}{k}(|(\beta_1 c_1 w)_x|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t &\leq 2\int_Q (s\theta)^3 \frac{\beta(x)^2}{k}|(\beta_1 c_1)_x|^2|w|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t \\ &+ 2\int_Q (s\theta)^3 \frac{\beta(x)^2}{k}|\beta_1 c_1|^2|w_x|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t \qquad (3.10)\end{aligned}$$

Under the assumptions on the coefficients, we have

$$2\int_Q (s\theta)^3 \frac{\beta(x)^2}{k}|(\beta_1 c_1)_x|^2|w|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t \leq C\int_Q (s\theta)^3 \frac{x^2}{k}|w|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t. \qquad (3.11)$$

And so for s large enough, we can absorb $2\int_Q (s\theta)^3 \frac{\beta(x)^2}{k}|(\beta_1 c_1)_x|^2|w|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t$ by by left term of inequality (3.8). Which means that there is a constant ε such that for s very large, we have

$$2\int_Q (s\theta)^3 \frac{\beta(x)^2}{k}|(\beta_1 c_1)_x|^2|w|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t \leq \varepsilon\int_Q s^3\theta^3 \frac{x^2}{k}|w|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t. \qquad (3.12)$$

Like $|\beta_1 c_1| \leq C$ and $\beta(x)^2 \leq Cx^2$, we also have

$$I_2 = 2\int_Q (s\theta)^3 \frac{\beta(x)^2}{k}|\beta_1 c_1|^2|w_x|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t \leq C\int_Q (s\theta)^3 \frac{x^2}{k}|w_x|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t. \qquad (3.13)$$

Let $\delta > 0$ be small. On $[0,\delta]$, $e^{2s\Phi} \leq e^{-cs\theta(t)}$. What gives

$$\int_0^\delta (s\theta)^2 \frac{x^2}{k^2} e^{2s\Phi} dx \leq (s\theta)^2 e^{-cs\theta(t)} \int_0^\delta \frac{x^2}{k^2} dx.$$

Under the hypothesis $xk'(x) \leq \lambda k(x)$, we have $k(x) \geq Cx^\lambda$ close to 0, so $\frac{x^2}{k^2} \leq Cx^{2-2\lambda}$, and $\int_0^\delta x^{2-2\lambda}dx < \infty$ because $\lambda < 1$.

As a result

$$\int_0^\delta (s\theta)^2 \frac{x^2}{k^2} e^{2s\Phi} dx \leq C(s\theta)^2 e^{-cs\theta(t)}.$$

For s large, $(s\theta)^2 e^{-cs\theta(t)} \to 0$ uniformly in $t \in [\varepsilon, T-\varepsilon]$, and near $t = 0, T$, $\theta(t)$ is very large, so even more decay.

On $[\delta, 1]$, $k(x) \geq k(\delta) > 0$, so $\frac{x^2}{k^2} \leq C$, and $(s\theta)^2$ is bounded by a constant times s^2.

We therefore have:

$$I_2 \leq \left(\sup_{x,t}(s\theta)^2 \frac{x^2}{k^2} e^{2s\Phi} \cdot e^{-2s\Phi}\right)\int_Q s\theta\, k|w_x|^2 e^{2s\Phi}.$$

But $\sup_{x,t}(s\theta)^2 \frac{x^2}{k^2} e^{2s\Phi}$ is finite for fixed s, and in fact for large s, the exponential decay near $x = 0$ causes this supremum to tend towards 0.

As result, we can also absorb $2\int_Q (s\theta)^3 \frac{\beta(x)^2}{k}|\beta_1 c_1|^2|w_x|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t$ by left term of inequality (3.8).

That is to say that it exists a constant ε such that for s very large, we find

$$2\int_Q (s\theta)^3 \frac{\beta(x)^2}{k}|\beta_1 c_1|^2|w_x|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t \leq \varepsilon \int_Q (s\theta)^3 \frac{x^2}{k}|w_x|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t. \tag{3.14}$$

The same arguments allow us to say that we can absorb the term as follows.
Finally, by combining inequalities (3.12) and (3.14) we find

$$\int_Q (s\theta)^3 \frac{\beta(x)^2}{k}|(\beta_1 c_1 w)_x|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t \leq \varepsilon \int_Q s\theta k|w_x|^2 e^{2s\Phi} + \varepsilon \int_Q s^3\theta^3 \frac{x^2}{k}|w|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t. \tag{3.15}$$

The same arguments allow us to say that we can absorb the term as follows

$$\int_Q (s\theta)^3 \frac{\beta(x)^2}{k(x)}|(\beta_2 c_2 v)_x|^2) e^{2s\Phi}\mathrm{d}x\mathrm{d}t \leq \varepsilon \int_Q s\theta k|v_x|^2 e^{2s\Phi} + \varepsilon \int_Q s^3\theta^3 \frac{x^2}{k}|v|^2 e^{2s\Phi}\mathrm{d}x\mathrm{d}t. \tag{3.16}$$

From the inequalities (3.15) and (3.16), we have

$$\begin{aligned} &\int_Q \left(s\theta k(x)(|v_x|^2 + |w_x|^2) + s^3\theta^3 \frac{x^2}{k(x)}(|v|^2 + |w|^2) \right) e^{2s\Phi}\mathrm{d}x\mathrm{d}t \\ \leq \quad & C\int_Q (|w|^2 + |v|^2)\mathrm{d}x\mathrm{d}t \end{aligned} \tag{3.17}$$

Let $\omega' \subset\subset \omega \cap \mathcal{O}$, we define and the function $\zeta \in \mathcal{C}^\infty(\Omega)$ as follows

$$\begin{cases} 0 \leq \zeta(x) \leq 1 & \forall x \in \Omega \\ \zeta(x) = 1 \quad \text{if} & x \in \omega' \\ \zeta(x) = 0 \quad \text{if} & x \in \Omega \setminus \omega' \end{cases}$$

In addition it is assumed that

$$\frac{\nabla\zeta}{\zeta^{1/2}} \in L^\infty(\Omega) \quad \text{and} \quad \frac{\Delta\zeta}{\zeta^{1/2}} \in L^\infty(\Omega). \tag{3.18}$$

Indeed, just take $\zeta = \zeta_0^4$ where $\zeta_0 \in \mathcal{C}^\infty(\Omega)$, to satisfy the two previous conditions on ζ.
To upper bound $\int_Q (|w|^2 + |v|^2\mathrm{d}x\mathrm{d}t$ by $C\int_0^T\int_{\omega'}|v|^2\mathrm{d}x\mathrm{d}t$ using the function ζ and the properties of the adjoint system.
Multiplying by ζv and by ζw and integrating over Q the first and second equations of system 2, we find

$$\begin{cases} -\int_Q v_t\zeta v - \int_Q (kv_x)_x\zeta v + \int_Q b_2\zeta v^2 - \int_Q (\beta_2 c_2 v)_x\zeta v = 0 \\ \\ -\int_Q w_t\zeta w - \int_Q (k(x)w_x)_x\zeta w + \int_Q b_1\zeta w^2 - \int_Q (\beta_1 c_1 w)_x\zeta w = \int_Q v 1_{\mathcal{O}}\zeta w. \end{cases} \tag{3.19}$$

The calculation and use of hypotheses 1.4 of each term of system (2.4) lead to the following estimates.

Integration by parts in x.

We obtain:

$$\int_Q k\zeta(v_x)^2 = -\int_Q k\zeta_x v v_x - \int_Q b_2\zeta v^2 - \int_Q \beta_2 c_2 \zeta_x v^2 - \int_Q \beta_2 c_2 \zeta v v_x.$$

Let us right-majorize each term and choosing ε small and absorbing, we obtain

$$\int_Q \zeta v^2 \mathrm{d}x\mathrm{d}t \leq C\int_Q \zeta(v_x)^2 \mathrm{d}x\mathrm{d}t \leq C\int_0^T\int_{\omega'} v^2 \mathrm{d}x\mathrm{d}t.$$

As a result

$$\int_Q \zeta(v^2+(v_x)^2)\mathrm{d}x\mathrm{d}t \leq C\int_0^T\int_{\omega'} v^2\mathrm{d}x\mathrm{d}t. \tag{3.20}$$

We obtain

$$\begin{aligned}\int_Q k\zeta(w_x)^2\mathrm{d}x\mathrm{d}t &= -\int_Q k\zeta_x w w_x \mathrm{d}x\mathrm{d}t - \int_Q b_1\zeta w^2\mathrm{d}x\mathrm{d}t - \int_Q \beta_1 c_1 \zeta_x w^2 \mathrm{d}x\mathrm{d}t\\ &- \int_Q \beta_1 c_1 \zeta w w_x \mathrm{d}x\mathrm{d}t + \int_Q v 1_{\mathcal{O}}\zeta w \mathrm{d}x\mathrm{d}t.\end{aligned}$$

As for v, we major the terms in w_x by $\varepsilon\int k\zeta(w_x)^2 + C_\varepsilon\int_0^T\int_{\omega'} w^2\mathrm{d}x\mathrm{d}t$, and the terms in w^2 by $C\int_0^T\int_{\omega'} w^2\mathrm{d}x\mathrm{d}t$.

The term $\int_Q v1_{\mathcal{O}}\zeta w\mathrm{d}x\mathrm{d}t$ is increased by $C\int_0^T\int_{\omega'}(v^2+w^2)\mathrm{d}x\mathrm{d}t$.

What gives

$$\int_Q k\zeta(w_x)^2\mathrm{d}x\mathrm{d}t \leq \varepsilon\int_Q k\zeta(w_x)^2 + C\int_0^T\int_{\omega'}(v^2+w^2)\mathrm{d}x\mathrm{d}t.$$

Hence

$$\int_Q k\zeta(w_x)^2\mathrm{d}x\mathrm{d}t \leq C\int_0^T\int_{\omega'}(v^2+w^2)\mathrm{d}x\mathrm{d}t.$$

From Poincaré's inequality we obtain

$$\int_Q \zeta w^2 \leq C\int_Q \zeta(w_x)^2\mathrm{d}x\mathrm{d}t \leq C\int_0^T\int_{\omega'}(v^2+w^2)\mathrm{d}x\mathrm{d}t. \tag{3.21}$$

Let's use a uniqueness argument. Suppose that $v\equiv 0$ on $\omega'\times(0,T)$.

Then (3.20) gives $\int_Q \zeta(v^2+(v_x)^2) = 0$, so $v\equiv 0$ on the support of ζ, so on a neighborhood of ω'.

Also (3.21) leads to $\int_Q \zeta w^2 \leq C\int_{\omega'} w^2$. But if $v\equiv 0$ on ω', the equation of w becomes

$$-w_t - (kw_x)_x + b_1 w - (\beta_1 c_1 w)_x = 0 \quad \text{on } \omega'\times(0,T).$$

By uniqueness of the Cauchy problem, $w \equiv 0$ on Q.

Thus, $w \equiv v \equiv 0$ on Q.

Now consider the continuous quadratic application

$$q : L^2(\Omega)^2 \longrightarrow \mathbb{R}^+,\ (w_T, v_T) \mapsto (w, v) \mapsto \int_0^T \int_{\omega'} |v|^2 \mathrm{d}x\mathrm{d}t, \tag{3.22}$$

where (w, v) is the solution of the adjoint system (2.4).

The uniqueness argument shows that if $q(w_T, v_T) = 0$ implies $(w_T, v_T) = 0$. By a compactness lemma (see e.g. [17]), we obtain the following inequality

$$\|(w_T, v_T)\|^2_{(L^2(\Omega))^2} \leq C \int_0^T \int_{\omega'} |v|^2 \mathrm{d}x\mathrm{d}t.$$

By standard energy estimation for the retrograde system, we have

$$\int_Q (|w|^2 + |v|^2)\mathrm{d}x\mathrm{d}t \leq C\|(w_T, v_T)\|_{L^2(\Omega)^2} \mathrm{d}x\mathrm{d}t.$$

Finally

$$\int_Q (|w|^2 + |v|^2) \leq C \int_0^T \int_{\omega'} |v|^2 \mathrm{d}x\mathrm{d}t. \tag{3.23}$$

In this section we prove our main result Theorem 1.3.

First, as consequence of Carleman inequality, we give a sketch of the proof Proposition 3.2. As a consequence of this result, we state the null controllability of linear degenerate parabolic equations but we do not give its prove since nowadays it is classical. See e.g. [13, 18, 11] for classical results. Finally we conclude the section with the non linear result.

Sketch of the proof of Proposition 3.2. To prove Proposition 3.2 we proceed as in [5].

Multiplying the second equation of the system (2.4) by v and integrating on $(0, 1)$, we obtain

$$\begin{aligned} \frac{d}{2dt}\int_\Omega v^2(x,t)\mathrm{d}x \quad &- \quad \int_\Omega k(x)v_x^2(x,t)\mathrm{d}x - \int_\Omega b_2(x,t)v(x,t)\mathrm{d}x \\ &- \quad \int_\Omega \beta_2(x)c_2(x,t)v(x,t)v_x(x,t)\mathrm{d}x = 0 \end{aligned}$$

Let us observe that according to (1.5), we have, for all $x \in [0, 1]$,

$$k(x) \geq k(1)\beta_2^2(x)C(\beta_2), \qquad \text{for some constant } C(\beta_2) > 0. \tag{3.24}$$

Then,

$$\begin{aligned} \int_\Omega k(x)v_x^2(x,t)\mathrm{d}x \quad &= \quad \frac{d}{2dt}\int_\Omega v^2(x,t)\mathrm{d}x - \int_\Omega b_2(x,t)v^2(x,t)\mathrm{d}x \\ &- \quad \int_\Omega (2k(1)C(\beta_2)\beta_2^2(x))^{1/2}\frac{c_2(x,t)}{(2k(1)C(\beta_2)\beta_2^2(x))^{1/2}}v(x,t)v_x(x,t)\mathrm{d}x \\ &\leq \quad \frac{d}{2dt}\int_\Omega v^2(x,t)\mathrm{d}x + \|b_2\|_\infty \int_\Omega v^2(x,t)\mathrm{d}x + \frac{\|c_2\|_\infty^2}{4k(1)C(\beta_2)}\int_\Omega v(x,t)^2\mathrm{d}x \\ &+ \quad k(1)C(\beta_2)\int_\Omega \beta_2^2(x)v_x^2(x,t)\mathrm{d}x \quad \forall t \in [0, T]. \end{aligned}$$

From (3.24), we have

$$\begin{aligned} 0 \leq \int_\Omega (k(x) - k(1)C(\beta_2)\beta_2^2(x))v_x^2(x,t)\mathrm{d}x &\leq \frac{d}{2dt}\int_\Omega v^2(x,t)\mathrm{d}x + \|b_2\|_\infty \int_\Omega v^2(x,t)\mathrm{d}x \\ &+ \frac{\|c_2\|_\infty^2}{4k(1)C(\beta_2)}\int_\Omega v(x,t)^2\mathrm{d}x \quad \forall t \in [0,T]. \end{aligned}$$

Which gives

$$\begin{aligned} 0 &\leq \frac{d}{dt}\int_\Omega v^2(x,t)\mathrm{d}x + 2\|b_2\|_\infty \int_\Omega v^2(x,t)\mathrm{d}x + \frac{\|c_2\|_\infty^2}{2k(1)C(\beta_2)}\int_\Omega v(x,t)^2\mathrm{d}x \quad \forall t \in [0,T] \\ &\leq \frac{d}{dt}\int_\Omega v^2(x,t)\mathrm{d}x + \max\left(2\|b_2\|_\infty, \frac{\|c_2\|_\infty^2}{2k(1)C(\beta_2)}\right)\int_\Omega v(x,t)^2\mathrm{d}x \quad \forall t \in [0,T]. \end{aligned}$$

This implies

$$0 \leq e^{Kt}\left(\frac{d}{dt}\int_\Omega v^2(x,t)\mathrm{d}x + K\int_\Omega v(x,t)^2\mathrm{d}x\right) \quad \forall t \in [0,T],$$

where $K = \max\left(2\|b_2\|_\infty, \dfrac{\|c_2\|_\infty^2}{2k(1)C(\beta_2)}\right)$.
Hence,

$$\frac{d}{dt}\left(e^{Kt}\int_\Omega v^2(x,t)\mathrm{d}x\right) \geq 0 \quad \forall t \in [0,T].$$

Thus, the function $t \longmapsto e^{Kt}\displaystyle\int_\Omega v^2(x,t)\mathrm{d}x$ is increasing on $[0,T]$.
As result, from (3.1) give

$$\int_\Omega v^2(x,0)\mathrm{d}x \leq Ce^{KT}\int_0^T\int_{\omega'}|v(x,t|^2\mathrm{d}x\mathrm{d}t, \quad \forall t \in [0,T]. \tag{3.25}$$

Now, multiplying the first equation of the system (2.4) by w and integrating on $(0,1)$. Proceeding as before, we obtain

$$\begin{aligned} 0 &\leq \frac{d}{dt}\int_\Omega w^2(x,t)\mathrm{d}x + 2\|b_1\|_\infty \int_\Omega w^2(x,t)\mathrm{d}x + \frac{\|c_1\|_\infty^2}{2k(1)C(\beta_1)}\int_\Omega w(x,t)^2\mathrm{d}x + \int_\Omega w(x,t)^2\mathrm{d}x \\ &+ \int_\Omega v(x,t)^2\mathrm{d}x \quad \forall t \in [0,T] \\ &\leq \frac{d}{dt}\int_\Omega w^2(x,t)\mathrm{d}x + \max\left(1, 2\|b_1\|_\infty, \frac{\|c_1\|_\infty^2}{2k(1)C(\beta_1)}\right)\int_\Omega w(x,t)^2\mathrm{d}x + \int_\Omega v(x,t)^2\mathrm{d}x \quad \forall t \in [0,T]. \end{aligned}$$

This implies

$$0 \leq e^{K_1 t}\left(\frac{d}{dt}\int_\Omega w^2(x,t)\mathrm{d}x + K_1\int_\Omega w(x,t)^2\mathrm{d}x\right) + \int_\Omega v(x,t)^2\mathrm{d}x \quad \forall t \in [0,T],$$

where $K_1 = \max\left(1, 2\|b_1\|_\infty, \dfrac{\|c_1\|_\infty^2}{2k(1)C(\beta_1)}\right)$.

Hence,

$$-\frac{d}{dt}\left(e^{K_1 t}\int_\Omega k(x)w^2(x,t)\mathrm{d}x\right) \le Ce^{K_1 t}\int_0^T\int_{\omega'} v(x,\tau)|^2\mathrm{d}x\mathrm{d}\tau,\ \forall t\in[0,T].$$

Thus, for every $0 \le s \le t \le T/2$ and (3.23), we get

$$\int_\Omega k(x)w^2(x,s)\mathrm{d}x \le C\int_\Omega k(x)w^2(x,t)\mathrm{d}x + C\int_0^T\int_{\omega'}|v(x,\tau)|^2\mathrm{d}x\mathrm{d}\tau, \quad \forall t\in[0,T/2].$$

Integrating on $[T/4, T/2]$ with respect to the variable t, we get for all $s \le T/4$.
Taking s large enough, we find by using the inequalities (3.1) and (3.25) that

$$\int_\Omega w^2(x,s)\mathrm{d}x \quad \le \quad C\int_0^T\int_{\omega'}|v(x,t)|^2\mathrm{d}x\mathrm{d}t. \tag{3.26}$$

Combining this result with the inequality (3.26) for $s = 0$ and (3.25) we can conclude that

$$\int_\Omega \left(w^2(x,0) + v^2(x,0)\right)\mathrm{d}x \le C\int_0^T\int_{\omega'}|v(x,t)|^2\mathrm{d}x\mathrm{d}t. \tag{3.27}$$

This concludes the proof of Proposition 3.2. □

Observe that the observability inequality implies the null controllability of system

$$\begin{cases} y_t - (k(x)y_x)_x + b_1(x,t)y + \beta_1(x)c_1(x,t)y_x = h1_\omega & \text{in } Q,\\ z_t - (k(x)z_x)_x + b_2(x,t)z + \beta_2(x)c_2(x,t)z_x = y1_{\mathcal{O}} & \text{in } Q,\\ y(\sigma,t) = z(\sigma,t) = 0 & \text{on } \Sigma,\\ y(x,0) = y_0(x) \ \ z(x,0) = z_0(x) & \text{in } \Omega. \end{cases} \tag{3.28}$$

In fact, by a minimization argument (see [6, 2, 1]), it can be proven that

Theorem 3.6. *For $T > 0$ and $y_0, z_0 \in L^2(0,1)$. Let us assume (1.2) and (1.4). Then, system (2.1) is null controllable, that is, there exists $h \in L^2(\omega\times(0,T))$ such that the solution pair (y,z) of The system (2.1) satisfies $y(x,T) = z(x,T) = 0$ for all $x \in [0,1]$. And furthermore, there exists a constant $C > 0$ depending only on T, such that*

$$\int_0^1\int_\omega |\tilde{h}|^2\mathrm{d}x\mathrm{d}x \le C\left(\|y_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)}\right). \tag{3.29}$$

4 Null controllabilityof semilinear degenerate coupled equations

This section is devoted to the proof of the main theorem (Theorem 1.3).

Let $y_0, z_0 \in H^1_k(0,1)$. Let us assume that $f_1(x,t,\cdot,\cdot), f_2(x,t,\cdot,\cdot), g_1(x,t,\cdot,\cdot), g_2(x,t,\cdot,\cdot) \in \mathcal{C}(\mathbb{R}^2)$, $\forall (x,t) \in Q$.
For every $(\tilde{y}, \tilde{z}) \in X$, we consider the linear coupled system

$$\begin{cases} y_t - (k(x)y_x)_x + f_1(x,t,\tilde{y},\tilde{y}_x)y + \beta_1(x)\left(\dfrac{f_2(x,t,\tilde{y},\tilde{y}_x)}{\beta_1(x)}\right)y_x = h1_\omega & \text{in } Q, \\ z_t - (k(x)z_x)_x + g_1(x,t,\tilde{z},\tilde{z}_x)z + \beta_2(x)\left(\dfrac{g_2(x,t,\tilde{z},\tilde{z}_x)}{\beta_2(x)}\right)z_x = y1_{\mathcal{O}} & \text{in } Q, \\ y(\sigma,t) = z(\sigma,t) = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) \;\; z(x,0) = z_0(x) & \text{in } \Omega. \end{cases} \tag{4.1}$$

Observe that (4.1) is of the form (2.4) with

$$\begin{cases} b_1 = b_{\tilde{y}} = f_1(x,t,\tilde{y},\tilde{y}_x) \in L^\infty(Q), \quad c_1 = c_{\tilde{y}} = \dfrac{f_2(x,t,\tilde{y},\tilde{y}_x)}{\beta_1(x)} \in L^\infty(Q) \\ \\ b_2 = b_{\tilde{z}} = g_1(x,t,\tilde{z},\tilde{z}_x) \in L^\infty(Q), \quad c_2 = c_{\tilde{z}} = \dfrac{g_2(x,t,\tilde{z},\tilde{z}_x)}{\beta_2(x)} \in L^\infty(Q) \end{cases}$$

Using Theorem 3.6, there exist a control $\tilde{h} \in L^2(\omega \times (0,T))$ such that the solution $(y_{\tilde{h}}, z_{\tilde{h}})$ of the system (4.1) withe $h = \tilde{h}$ associated to the potentials $b_{\tilde{y}}$, $c_{\tilde{y}}$, $b_{\tilde{z}}$ and $c_{\tilde{z}}$ satisfies $y_{\tilde{h}}(x,T) = z_{\tilde{h}}(x,T) = 0$ for all $x \in [0,1]$. Furthermore, there exists a constant $C > 0$ depending only on T, such that

$$\|\tilde{h}\|^2 \leq C\left(\|y_0\|^2_{L^2(\Omega)} + \|z_0\|^2_{L^2(\Omega)}\right). \tag{4.2}$$

On the other hand, from Theorem 2.1, we obtain that

$$(y_{\tilde{h}}, z_{\tilde{h}}) \in X = (L^2(0,T;H^1_k(\Omega)))^2 \quad \text{and} \quad \|(y_{\tilde{h}}, z_{\tilde{h}})\|_X \leq C(\|(y_0,z_0)\|_{(H^1_k(0,1))^2} + \|\tilde{h}\|_{L(q)}). \tag{4.3}$$

For any $(\tilde{y}, \tilde{z}) \in X$, one defines the family of controls :

$$\mathcal{A}(\tilde{y},\tilde{z}) = \{h \in L^2(Q) \;: (y_h, z_h) \in X, y_h(x,T) = z_h(x,T) = 0 \; in \; \Omega \;\text{ and } h \text{ satisfies (4.2)}\}.$$

Thus, we can introduce the multi-valued mapping

$$\Lambda\;(\tilde{y},\tilde{z}) \in X \longmapsto \Lambda(\tilde{y},\tilde{z}) \subset X,$$

where

$$\Lambda(\tilde{y},\tilde{z}) = \{(y_h, z_h) \in X, \text{ solution of (4.1), satisfies (4.3) and } h \in \mathcal{A}(\tilde{y},\tilde{z})\}.$$

We will prove that this mapping admits at least one fixed point (y,z). Notice that Kakutani's fixed point theorem can be applied to Λ thus, ensuring the existence of (at least) one fixed point of Λ in X.

First, from Theorem 3.6 and the inequalities (4.2) and (4.3), we deduce that $\Lambda(\tilde{y},\tilde{z})$ is for every $(\tilde{y},\tilde{z})$ a nonempty and compact set. Moreover, $\Lambda(\tilde{y},\tilde{z})$ is a closed and convex subset for any $(\tilde{y},\tilde{z})$. Moreover, the mapping $(\tilde{y},\tilde{z}) \longmapsto \Lambda(\tilde{y},\tilde{z})$ is upper hemicontinuous (see [4, 5] for more details).

Now we assume now that $f_1(x,t,\cdot,\cdot), f_2(x,t,\cdot,\cdot), g_1(x,t,\cdot,\cdot), g_2(x,t,\cdot,\cdot)$ belong to $L^\infty(\mathbb{R}^2)$, $\forall (x,t) \in Q$. We introduce the functions $\rho_i(x,t,\cdot,\cdot) \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ such that $\rho_i(x,t,\cdot,\cdot) \geq 0$, and $\rho_i(x,t,\cdot,\cdot) \geq 0$ in $\mathbb{R}^2$, supp$\rho_i(x,t,\cdot,\cdot) \subset \overline{B}(0,1)$ and

$$\int_{\mathbb{R}^2} \rho_i(x,t,q,p)\mathrm{d}q\mathrm{d}p = 1, \quad \forall (x,t) \in Q \quad \text{ and } i = 1,2.$$

We consider the functions $\rho_{i,n}$, $f_{i,n}$ and $g_{i,n}$ for $i = 1,2$ and $n \geq 1$, with

$$\begin{cases} \rho_{i,n}(x,t,q,p) = \dfrac{1}{n^2}\rho_i(x,t,nq,np) \quad \forall (q,p) \in \mathbb{R}^2, \\ f_{i,n} = \rho_{i,n} * f_i \qquad g_{i,n} = \rho_{i,n} * g_i. \end{cases}$$

Then, it is not difficult to see that $f_{i,n}$ and $g_{i,n}$ satisfy:

$$\begin{cases} f_{i,n}(x,t,\cdot,\cdot),\ f_{i,n}(x,t,\cdot,\cdot),\ f_{i,n}(x,t,\cdot,\cdot),\ f_{i,n}(x,t,\cdot,\cdot) \in L^\infty(\mathbb{R}^2) \\ f_{i,n}(x,t,\cdot,\cdot) \longrightarrow f_i(x,t,\cdot,\cdot) \quad \text{ uniformly in } \mathbb{R}^2,\ \forall (x,t) \in Q, \\ g_{i,n}(x,t,\cdot,\cdot) \longrightarrow g_i(x,t,\cdot,\cdot) \quad \text{ uniformly in } \mathbb{R}^2,\ \forall (x,t) \in Q. \end{cases}$$

For each $n \geq 1$, we obtain from Theorem 3.6, a control $h_n \in L^2(\omega \times (0,T))$ such that the following system

$$\begin{cases} y_{t,n} - (k(x)y_{x,n})_x + F(x,t,y_n,y_{x,n}) = h_n 1_\omega & \text{in } Q, \\ z_{t,n} - (k(x)z_{x,n})_x + G(x,t,z_n,z_{x,n}) = y_n 1_{\mathcal{O}} & \text{in } Q, \\ y_n(\sigma,t) = z_n(\sigma,t) = 0 & \text{on } \Sigma, \\ y_n(x,0) = y_0(x) \ \ z_n(x,0) = z_0(x) & \text{in } \Omega, \end{cases} \tag{4.4}$$

admits a least solution $(y_n,z_n) \in X$ that verifies $y_n(x,T) = z_n(x,T) = 0$ for all $x \in [0,1]$ and $\|(y_n,z_n)\|_X \leq R$ and $\|h_n\|_{L^2(\omega\times(0,T))} \leq R, \quad \forall n \geq 1$.
We have that $(y_n,z_n) \in K$ for every $n \geq 1$, with K a compact subset of X. Therefore, we can assume that at least for a subsequence that we are going to denote (y_n,z_n) and h_n again respectively such that $(y_n,z_n) \longrightarrow (y,z)$ strongly in X and $h_n \longrightarrow h$ weakly in $L^2(Q)$. Passing to the limit in (4.4), we obtain a control $h \in L^2(\omega \times (0,T))$ such that it exists a control $h \in L^2(Q)$ such that the corresponding solution a solution (y,z) to (1.1) that satisfies (1.3). we can assume that at least for a subsequence

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